INTERNATIONAL MATHEMATICS TOURNAMENT OF TOWNS

Junior O-Level, Spring 2014.

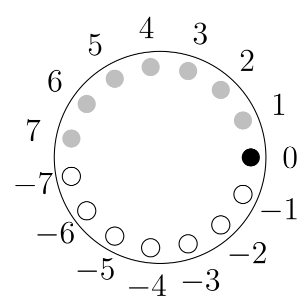
1. Each of given 100 numbers was increased by 1. Then each number was increased by 1 once more. Given that the first time the sum of the squares of the numbers was not changed find how this sum was changed the second time.

ANSWER. The sum increased by 200.

SOLUTION. Given that the sum of the squares did not change when we added 1 to each number, we have $(a_1+1)^2+(a_2+1)^2+\dots+(a_{100}+1)^2)-(a_1^2+a_2^2+\dots+a_{100}^2)=0$ or $(2a_1+1)+(2a_2+1)+\dots+(2a_{100}+1)=0$. Therefore, we have $a_1+a_2+\dots+a_{100}=-50$. If we increase each number by 1 once more, the sum of squares will change by $(a_1+2)^2+(a_2+2)^2+\dots+(a_{100}+2)^2-(a_1^2+a_2^2+\dots+a_{100}^2)=(4a_1+4)+(4a_2+4)+\dots+(4a_{100}+4)=4\times(-50)+400=200$.

2. Mother baked 15 pasties. She placed them on a round plate in a circular way: 7 with cabbage, 7 with meat and one with cherries in that exact order and put the plate into a microwave. All pasties look the same but Olga knows the order. However she doesn't know how the plate has been rotated in the microwave. She wants to eat a pasty with cherries. Can Olga eat her favourite pasty for sure if she is not allowed to try more than three other pasties?

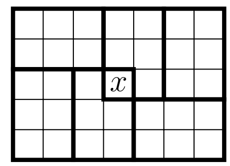
ANSWER. Yes, she can. SOLUTION. Denote the cherry pasty by 0, the cabbage pasties by 1,…, 7 and the meat pasties by $-1,\dots,-7$. If Olga does not get the cherry pasty on her first try, it must be either a cabbage pasty or a meat pasty. On her second try Olga takes the 4-th pasty from the first one in the direction to the cherry pasty. She gets either the cherry pasty 0, or the cabbage pasty 1,2,3, or the meat pasty $-1,-2,-3$.



On her last try Olga takes the second pasty from her second try in the direction to the cherry pasty and gets either the cherry pasty 0, or the cabbage pasty 1, or the meat pasty -1. Hence, after at most three tries Olga knows the position of the cherry pasty for sure.

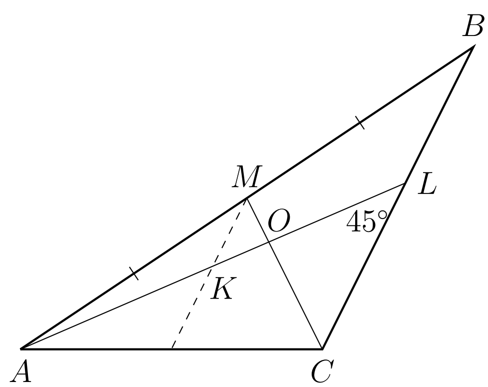
3. The entries of a 7×5 table are filled with numbers so that in each 2×3 rectangle (vertical or horizontal) the sum of numbers is 0. For 100 dollars Peter may choose any single entry and learn the number in it. What is the least amount of dollars he should spend in order to learn the total sum of numbers in the table for sure?

textscAnswer. 100. Solution. Let S be the total sum of the numbers in the table. Let Peter divide the table into 6 rectangles as shown on the picture (two rectangles overlap on a marked entry). Then $S = 0 \times 5 + (0 - x)$ where x is the value in the marked entry he would pay for. Thus it suffices to pay 100 dollars. Peter cannot find S for free because we may fill the table chesswise by a and $-a$ with arbitrary a, so the sum $S = -a$ canbe arbitrary.

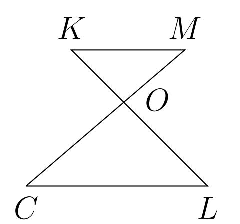


4. Point L is marked on side BC of triangle ABC so that AL is twice as long as the median CM. Given that angle ALC is equal to $45°$ prove that AL is perpendicular to CM.

Solution.



(a)



(b)

Let O be the point of intersection of AL and MC. Let K be the point of intersection of AL and the line drawn through M parallel to BC. Then $AK = KL$. Since MK is parallel to CL, triangles KMO and OLC are similar and we have $KO/(KL - KO) = MO/(MC - OM)$. Since $KL = MC$, we have $KO = OM$ and each of triangles OKM and OCL is isosceles and therefore $\angle OCL = \angle OLC = 45°$. Hence, $\angle COL = 90°$.

5. Ali Baba and the 40 thieves want to cross Bosporus strait. They made a line so that any two people standing next to each other are friends. Ali Baba is the first; he is also a friend with the thief next to his neighbour. There is a single boat that can carry 2 or 3 people and these people must be friends. A single person cannot sail. Can Ali Baba and the 40 thieves surely cross the strait?

textscAnswer. Yes, they can. SOLUTION. If the number of thieves n is even then A, T_1, T_2 (Alibaba and the two first thieves) sail to Europe, and A, T_1 sail back leaving T_2 in Europe. Then T_3, T_4 sail to Europe, T_2, T_3 sail back and now T_4 is in Europe and everybody else is in Asia. Continuing this process we end up with T_n (the last thief in line) in Europe, and $A, T_1, \ldots, T_{n-1}$ with the boat in Asia.

If the number of thieves n is odd then A, T_1, T_2 sail to Europe, and A, T_2 sail back leaving T_1 in Europe. Then T_2, T_3 sail to Europe, T_1, T_2 sail back and now T_3 is in Europe and everybody else is in Asia. Continuing this process we end up with T_n in Europe, and $A, T_1, \ldots, T_{n-1}$ with the boat in Asia.

We can see that after applying described operation the last thief in the line will be in Europe and all the remained gang in Asia. We are again in conditions of the original problem but the number of thieves decreased by 1. Therefore, we apply this process several times until we get only A, T_1, T_2 in Asia. Then the trio sail to Europe and join the gang.

INTERNATIONAL MATHEMATICS TOURNAMENT OF TOWNS

Senior O-Level, Spring 2014.

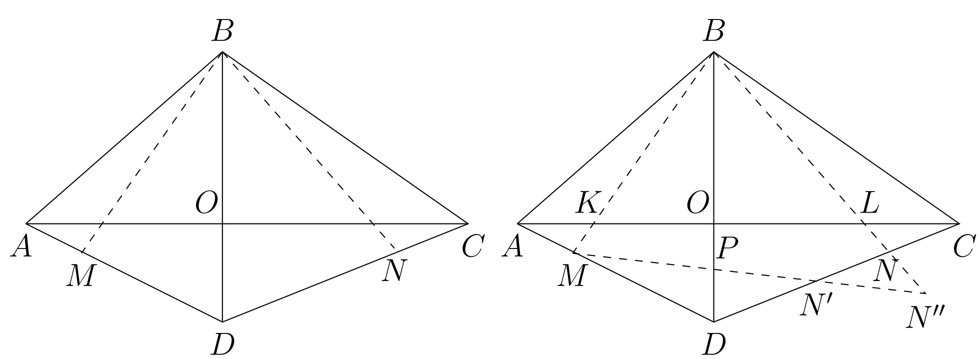
1. Inspector Gadget has 36 stones with masses 1 gram, 2 grams, ..., 36 grams. Doctor Claw has a superglue such that one drop of it glues two stones together (thus two drops glue 3 stones together and so on). Doctor Claw wants to glue some stones so that in obtained set Inspector Gadget cannot choose one or more stones with the total mass 37 grams. Find the least number of drops needed for Doctor Claw to fulfil his task.

ANSWER: 9.

SOLUTION. (a) Among the given stones there are 18 stones with odd masses which could be split into 9 pairs. To glue stones in pairs Doctor Claw needs 9 drops. In the new group of stones there is no stone with odd weight. Therefore, Inspector Gadget cannot fulfil his task.

(b) Let us split all stones into 18 pairs so that in each pair a total weight of stones is 37. Then Doctor Claw needs to "spoil" at least one stone in each pair which is impossible with less than 9 drops.

2. In a convex quadrilateral $ABCD$ the diagonals are perpendicular. Points M and N are marked on sides AD and CD respectively. Angles ABN and CBM are right angles. Prove that lines AC and MN are parallel.



(a) (b)

Solution 1. See Figure (b). Observe that $\angle BAC = \angle OBL$ and $\angle KBO = \angle BCA$. Then triangles KBO and OBC are similar, so $KO : OB = OB : OC$ and therefore $KO = OB^2/OC$. In similar way, $OL = OB^2/AO$. Hence

$$\frac{KO}{OL} = \frac{AO}{OC}. \tag{$*$}$$

Assume that MN is not parallel to AC. Through M draw a line parallel to AC and denote points P, N' and N'' on it as shown. Then triangles ADC and MDN' are similar and therefore $MP : PN' = AO : OC$. Comparing to $(*)$ we conclude that

$$\frac{KO}{OL} = \frac{MP}{PN'}. \tag{$**$}$$

Since triangles MBN'' and KBL are also similar, we have $KO : OL = MP : PN''$. Comparing to $(*)$ we conclude that $PN' = PN''$. A contradiction.

Solution 2. Introducing Cartesian coordinates one can assume that $A(a, 0)$, $B(0, b)$, $C(c, 0)$, $D(0, d)$ with $a < 0, b > 0, c > 0, d < 0$. Then MB is given by equation $cx - b(y - b) = 0$, and AD is given by equation $x/a + y/d = 1$. Solving the system we find y-coordinate of M: $y_M = (ac + b^2)d/(ac + bd)$. Permuting a and c we find $y_N = y_M$ which implies that $MN \parallel AC$.

3. Ali Baba and the 40 thieves want to cross Bosporus strait. They made a line so that any two people standing next to each other are friends. Ali Baba is the first; he is also a friend with the thief next to his neighbour. There is a single boat that can carry 2 or 3 people and these people must be friends. A single person cannot sail. Can Ali Baba and the 40 thieves surely cross the strait?

Solution. Let n be the number of the thieves (not counting Ali Baba). We will prove by induction that the gang can cross the strait. For $n = 1$ and 2 the base is obvious, one can check it for $n = 3$ as well. For simplicity of explanation we assume that they are going from Asia to Europe.

Assume that for any number $k = 1, 2, \dots, n$ our statement holds. Denote Ali-Baba by A and the thieves by $T_1, \dots, T_{n+1}$. First let $A, T_1, \dots, T_{n-1}$ cross the strait leaving T_n, T_{n+1} behind in Asia (it can be done according to the induction hypotheses).

Next $A, T_1, \dots, T_{n-2}$ sail back leaving T_{n-1} behind in Europe (again it can be done according to the induction hypotheses). Next T_n, T_{n+1} sail to Europe and then T_{n-1}, T_n go back bringing boat to Asia. Now $A, T_1, \dots, T_n$ are in Asia, so they cross the strait and join T_{n+1} .

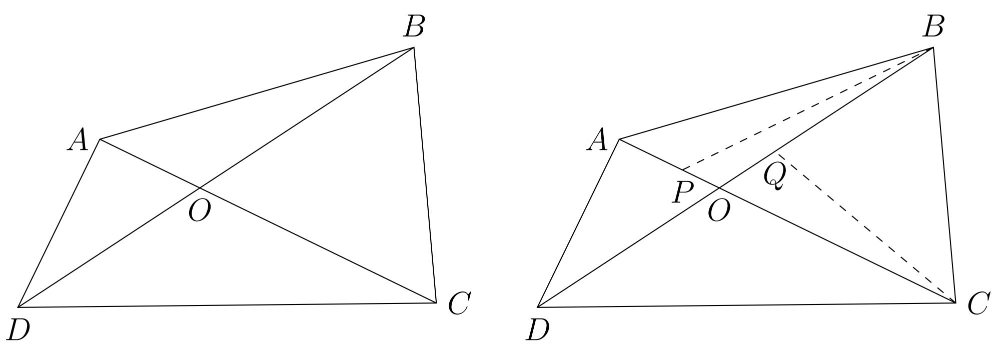
4. Positive integers a, b, c, d are pairwise coprime and satisfy the equation

$$ab + cd = ac - 10bd.$$

Prove that one can always choose three numbers among them such that one number equals the sum of two others.

SOLUTION. Rewriting the equation in the form $a(c-b) = (10b+c)d$ and given that a and d are coprime we conclude that $(c-b)$ is positive and divisible by d: $c = b + dx$ with x positive integer. Plugging c into latter equation and simplifying we get $(a-d)x = 11b$. Since c and b are coprime, x and b are also coprime and therefore either $x = 1$ or $x = 11$. In the former case we have $c = b + d$ and in the latter case $a = b + d$.

5. Park's paths go along sides and diagonals of the convex quadrilateral $ABCD$. Alex starts at A and hikes along $AB - BC - CD$. Ben hikes along AC; he leaves A simultaneously with Alex and arrives to C simultaneously with Alex. Chris hikes along BD; he leaves B at the same time as Alex passes B and arrives to D simultaneously with Alex. Can it happen that Ben and Chris arrive at point O of intersection of AC and BD at the same time? The speeds of the hikers are constant.



(a) (b)

SOLUTION Let v_A, v_B, v_C be the speeds of Alex, Ben and Chris respectively. Let Alex and Ben start hiking at time 0. By triangle inequality $AB + BC > AC$, Alex traveled a further distance than Ben in the same time interval as they started and finished simultaneously. Hence $v_A > v_B$. Similarly $BC + CD > BD$, so Alex traveled a further distance than Chris in the same time interval. Hence $v_A > v_C$.

Assume that Ben and Chris arrive to O at the same time. Then we can replace them with a single person Mikey who travels from B to O with speed v_B and from O to C with speed v_C while Alex travels from B to C in the same time interval. Mikey's speed is always less than Alex's speed but Mikey travels distance $BO + OC$ which is greater than BC. This is impossible. Therefore, Ben and Chris cannot arrive at O at the same time.

SOLUTION 2 See Figure (b). Let P be a point where Ben was when both Alex and Chris were in B. Since speeds of Alex and Ben are constant and Ben arrive to C at the same time as Alex, we have $AB : BC = AP : PC$ and therefore BP is a bisector of $\angle ABC$.

Let Q be a point where Chris was when Alex and Ben arrived in C. Similarly $BQ : QD = BC : BD$ so CQ is a bisector of $\angle BCD$. Observe that if Ben and Chris arrive at O simultaneously then P belongs to AO and Q belongs to OD.

Assume that Ben and Chris arrived to O simultaneously. Then $PO : OC = BO : OQ$ and since $\angle POB = \angle COQ$ we conclude that triangles BOP and COQ are similar. Then $\angle PBO = \angle OQC$, $\angle PBC + \angle BCQ = 180°$ and since BP and CQ are bisectors, $\angle ABC + \angle BCD = 360°$ which is impossible.