INTERNATIONAL MATHEMATICS TOURNAMENT OF TOWNS
Junior A-Level Paper, Spring 2014.

1. During Christmas party Santa handed out to the children 47 chocolates and 74 marmalades. Each girl got 1 more chocolate than each boy but each boy got 1 more marmalade than each girl. What was the number of the children?

Answer. 11 or 121. Solution. Each child got the same number of treats and the total number of treats is $74 + 47 = 121$. Therefore there could be either (a) 11 children, or (b) 121, or (c) just 1 child. The last is obviously impossible. Thus each child got 11, or 1 treat.

Remark. In case (a) let x denote the number of boys and c the number of chocolates each girl got. Then $(c-1)x + c(11-x) = 47$ or $11c = 47 + x$. The only integer solution with $0 \leq x \leq 11$ is $x = 8$, $c = 5$ (so, 8 boys, 3 girls). In case (b) each boy got just 1 marmalade, and each girl got just 1 chocolate (so, 74 boys and 47 girls).

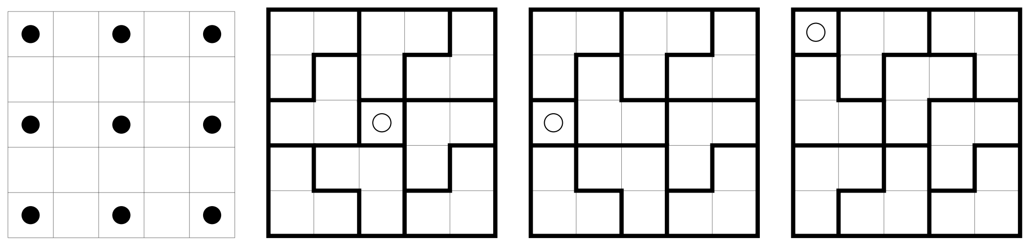
2. Peter marks several cells on a 5×5 board. Basil wins if he can cover all marked cells with three-cell corners. The corners must be inside the board and not overlap. What is the least number of cells Peter should mark to prevent Basil from winning? (Cells of the corners must coincide with the cells of the board).

Answer. 9 cells. Solution. If Peter marks 9 points as shown on (a) Basil cannot cover them. Indeed, no corner can cover more than one marked cell, so Basil needs 9 corners; but they contain 27 cells while the whole board contains only 25.

If Peter marks 8 cells Basil can cover all of them. Indeed, one of the cells shown on (a) is not marked. However the remaining 24 cells could be covered as shown on (b)–(d).

3. A square table is covered with a square cloth (may be of a different size) without folds and wrinkles. All corners of the table are left uncovered and all four hanging parts are triangular. Given that two adjacent hanging parts are equal prove that two other parts are also equal.

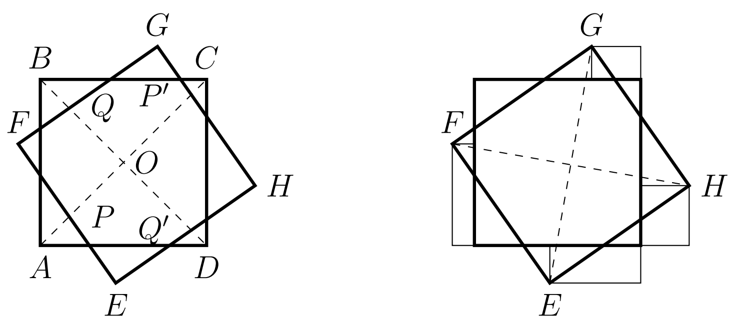
Solution 1. Let $ABCD$ be a cloth and $EFGH$ be a table (see Figure (a)). We see four hanging parts of the cloth and four triangular parts of the table which are not covered. Observe that all eight triangles are similar.



(a) (b) (c) (d)

Let us draw diagonals in $ABCD$. Observe that they are bisectors of the corresponding angles. Observe that since angles between AC and $FG \parallel EH$ and BD and $FE \parallel EH$ are equal and distances between two pairs of parallel lines are also equal then $QQ' = PP'$.

If triangles A and B are equal then their bisectors AP and BQ are equal and since $AO = BO = CO = DO$ we see that $PO = QO$. But then $P'O = Q'O$ and $P'C = Q'D$. Then triangles C and D are also equal.



(a) (b)

Solution 2 (see Figure (b)). We define the *weight* of the hanging triangle as its height dropped from the right corner. Obviously all hanging parts are similar. Therefore parts are equal if and only if their heights are equal. Therefore it is sufficient to prove that the the sums of weights of opposite parts are equal. Adding to these sums the side of the table we get projection of diagonal FH to the "horizontal" side of the the table and of diagonal EG to the "vertical" side of the the table. Since diagonals are equal and orthogonal and the sides of the table are orthogonal, we conclude that projections are equal.

4. The King called two wizards. He ordered First Wizard to write down 100 positive integers (not necessarily distinct) on cards without revealing them to Second Wizard. Second Wizard must correctly determine all these integers, otherwise both wizards will lose their heads. First Wizard is allowed to provide Second Wizard with a list of distinct integers, each of which is either one of the integers on the cards or a sum of some of these integers. He is not allowed to tell which integers are on the cards and which integers are their sums. Finally the King tears as many hairs from each wizard's beard as the number of integers in the list given to Second Wizard. What is the minimal number of hairs each wizard should lose to stay alive?

Solution. See solution to Senior **??**

5. There are several white and black points. Every white point is connected with every black point by a segment. Each segment is equipped with a positive integer. For any closed circuit the product of the numbers on the segments passed in the direction from white to black point is equal to the product of the numbers on the segments passed in the opposite direction. Can one always place the numbers at each point so that the number on each segment is the product of the numbers at its ends?

Solution. Let us denote white points W_j, $j = 1, 2, \dots, m$ and black points B_k, $k = 1, 2, \dots . n$. Let c_{jk} be a label on the segment from white point W_j to black point B_k. Consider closed circuit $W_1 - B_1 - W_j - B_k - W_1$. Then $c_{11}c_{jk} = c_{j1}c_{1k}$ and therefore $c_{jk} = c_{j1}c_{1k}/c_{11} = w_j b_k$ where $w_j = c_{j1}/d$, $b_k = c_{1k}d/c_{11}$, $d = \gcd(c_{11}, c_{21}, \dots, c_{m1})$. Obviously $w_1, \dots, w_m$ are integers and coprime.

Since $w_j b_k$ are integers and $w_1, \dots, w_m$ are coprime, then b_k are integers as well. Indeed, let b_k be not an integer, then represent it as irreducible ratio $b_k = b'/r$ with $r \geq 2$. Since $w_j b_k = w_j b'/r$ are integers r must divide w_j for all j which is impossible as these numbers are coprime.

6. A $3 \times 3 \times 3$ cube is made of $1 \times 1 \times 1$ cubes. What is the maximal number of small cubes one can remove so that the remaining solid has the following features:

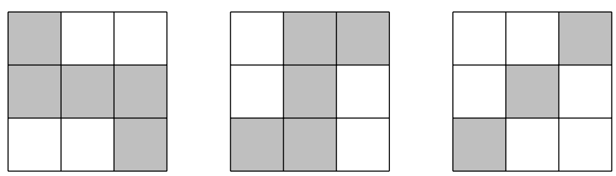
1) Projection of this solid on each face of the original cube is a 3×3 square;

2) The resulting solid remains face-connected (from each small cube one can reach any other small cube along a chain of consecutive cubes with common faces).

Answer: 14 small cubes.

Solution. Consider example with removed 14 cubes (remaining 13 cubes are shaded on these 3 layers). Each layer has cubes in each row and column; imposing layers we get a full square. Therefore the first condition is fulfilled. Top and middle layers are glued together through their central cubes. Each cube of the bottom layer is glued to the corresponding cube of the middle layer.

To prove that no more than 14 cubes could be removed we prove an estimate for the number n of remaining cubes. We can see from all 6 directions $6 \cdot 9$ of their faces. For connectivity one needs at least $(n-1)$ gluings ; therefore we do not see at least $2(n-1)$ faces, while the total number of faces is $6n$. Then $6n \geq 2(n-1) + 54$ and therefore $n \geq 13$.

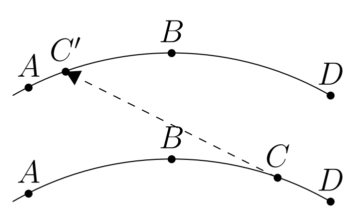


(a) Top (b) Middle (c) Bottom

7. Points A_1, A_2, …, A_{10} are marked on a circle clockwise. It is known that these points can be divided into pairs of points symmetric with respect to the centre of the circle. Initially at each marked point there was a grasshopper. Every minute one of the grasshoppers jumps over its neighbour along the circle so that the resulting distance between them doesn't change. It is not allowed to jump over any other grasshopper and to land at a point already occupied. It occurred that at some moment nine grasshoppers were found at points A_1, A_2, …, A_9 and the tenth grasshopper was on arc $A_9A_{10}A_1$. Is it necessarily true that this grasshopper was exactly at point A_{10}?

Answer. Yes.

Solution. 10 grasshoppers divide circle into 10 arcs. Let us paint them alternatively black and white. Originally sums of the lengths of white and black arcs are equal because for any arc the arc symmetric to it with respect to the center is of another colour.



It follows from the figure that the grasshopper's jump does not change these sums. Indeed, sum of arcs AC' and BD equals to the sum of arcs AB and CD. In the final configuration we know 4 black arcs and know where the fifth is located and therefore position of 10th grasshopper is defined uniquely. On the other hand, A_{10} satisfies to the condition that the sums of black and white arcs are equal.

INTERNATIONAL MATHEMATICS TOURNAMENT OF TOWNS
Senior O-Level Paper, Spring 2014.

1. Doono wrote several **1**s, placed signs "+" or "×" between every two of them, put several brackets and got 2014 in the result. His friend Dunno replaced all "+" by "×" and all "×" by "+" and also got 2014. Can this be true?

SOLUTION. Yes, it could be true. For example, consider the following expression consisting of 4027 **1**s:

$$1+\underbrace{1\times 1+1\times 1+\ldots+1\times 1}_{\text{2013 terms}}$$

which obviously equals 2014. After Doono changed signs it became

$$\underbrace{1\times 1+1\times 1+\ldots+1\times 1}_{\text{2013 terms}}+1$$

which also equals 2014.

2. Is it true that any convex polygon can be dissected by a straight line into two polygons with equal perimeters and

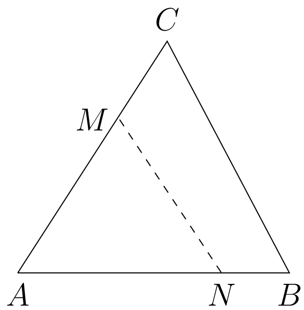
(a) equal greatest sides?

(b) equal smallest sides?

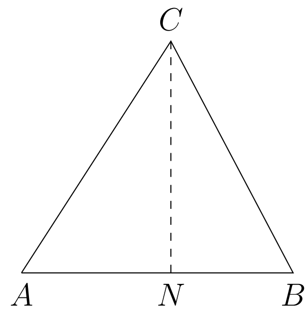
(a) ANSWER: Yes.

SOLUTION. Consider a convex polygon polygon and point M on its boundary. Consider its opposite point $N = N(M)$. It means that MN dissects polygon into two $MA_1 \ldots A_m N$ and $NA_{m+1} \ldots A_n M$ with equal perimeters (it is possible that M and N are among vertices of the original polygon). Here $MA_1 \ldots A_m N$ is in the counterclockwise direction. Define $f(M)$ as the greatest side of $MA_1 \ldots A_m N$. Observe that $f(M)$ continuously depends on M. Then $g(M) = f(M) - f(N(M))$ also continuously depends on M. However as M changes from original point M_0 to its opposite point N_0, $g(M)$ changes from $g(M_0)$ to $g(N_0) = -g(M_0)$. Therefore $g(M) = 0$ for some M.

Remark. $h(M)$ as the smallest side of $MA_1 \ldots A_m N$ is not continuous and these arguments do not work for Part (b).



(a)



(b)

(b) Answer: No.

Consider triangle ABC. We call points M and N opposite if (as on the figure (a)) $MA + AN = p/2$ where $p = a + b + c$ is a perimeter of ABC.

Consider first an equilateral triangle with sides $a = b = c$. We claim that the smallest cut between opposite points has the length $3a/4$. Indeed, The shortest cut MN must be orthogonal to bisector of angle CAB. To prove this, take another cut $M'N'$ with endpoints on the same sides and shift it so that M' coincides with M (and N' goes to some N''). Then we get triangle MNN'' such that $\angle MNN'' = 90^\circ$, thus $M'N' > MN$.

Hence in an equilateral triangle $MN \geq 3a/4$ and $MC + NB = a/2$, $AM = NB + a/2$, $AN = CM + a/2$ and therefore for equilateral triangle the answer is negative unless the cut passes through one of the vertices. The same is true for all triangles sufficiently close to equilateral.

Consider $M = C$. But then in CAN and CNB the smallest sides are $AN = p/2 - b = (a-b+c)/2$ and $NB = (a+b-c)/2$ where $a = BC$, $b = AC$ and $c = AB$ and $AN \neq NB$ if $b \neq c$.

Therefore one cannot dissect any triangle which is close to equilateral but has all sides different.

3. The King called two wizards. He ordered First Wizard to write down 100 positive real numbers (not necessarily distinct) on cards without revealing them to Second Wizard. Second Wizard must correctly determine all these numbers, otherwise both wizards will lose their heads. First Wizard is allowed to provide Second Wizard with a list of distinct numbers, each of which is either one of the numbers on the cards or a sum of some of these numbers. He is not allowed to tell which numbers are on the cards and which numbers are their sums. Finally the King tears as many hairs from each wizard's beard as the number of numbers in the list given to Second Wizard. What is the minimal number of hairs each wizard should lose to stay alive?

ANSWER. 101

SOLUTION The first wizard writes $1, 2, 4, \dots, 2^{99}$ and lists all these numbers and their sum $2^{100} - 1$. Then the second wizard understands that there is a card with a number not exceeding 1, there is another card with a number not exceeding 2, ..., and there is 100th card with a number not exceeding 2^{99}. Then their sum does not exceed $2^{100} - 1$ and the equality is possible if and only if numbers are $1, 2, 4, \dots, 2^{99}$.

Now let us prove that 100 is too little. Suppose the second wizard has received 100 distinct numbers, and $a < b$ are two of them. Then it is possible that all these numbers are from the cards. But if b is replaced by $b - a$ then the list also fits.

4. In the plane are marked all points with integer coordinates (x, y), $0 \leq y \leq 10$. Consider a polynomial of degree 20 with integer coefficients. Find the maximal possible number of marked points which can lie on its graph.

SOLUTION (Michael Chow) We need to consider integer solutions of the system of inequalities:

$$0 \leq P(x) \leq 10. \qquad (*)$$

Let us prove by contradiction that there are no more than 20 integer solutions to (**??**). Assume that $x_1 < x_2 < \ldots < x_{21}$ satisfy (**??**); denote $a = x_1$, $b = x_{21}$; then $b - a \geq 20$.

Consider $P(b) - P(a)$; since both a, b satisfy (**??**) we conclude that $|P(b) - P(a)| \leq 10$. However since $P(x)$ has integer coefficient, the number $P(b) - P(a)$ must be divisible by $(b - a)$ (indeed, $P(b) - P(a) = (b - a)R(a, b)$ where R is a polynomial with integer coefficients). Since $|P(b) - P(a)| \leq 10$ and $b - a \geq 20$ divisibility implies that $P(b) - P(a) = 0$. So $P(a) = P(b) = c$ with $0 \leq c \leq 10$.

Then $P(x) = (x - a)(b - x)Q(x) + c$ where $Q(x)$ is a polynomial of degree 18. Observe that $(x - a)(b - x) \geq 19$ for integer $x = a + 1, \dots, b - 1$. Then $P(x)$ cannot satisfy (**??**) unless $Q(x) = 0$. Indeed, if $Q(x) \neq 0$ then either $P(x) \leq -19 + c < 0$ or $P(x) \geq 19 + c > 10$.

Therefore $Q(x_k) = 0$, $k = 2, \dots, 20$ but polynomial $Q(x)$ of degree 18 cannot have more than 18 roots. A contradiction.

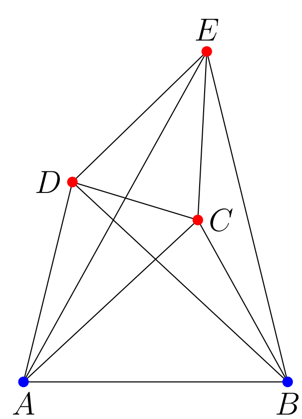
On the other hand, for $P(x) = (x - x_1)(x - x_2) \cdots (x - x_{20})$ the system (**??**) has 20 solutions $x_1, \dots, x_{20}$.

SOLUTION 2. We need to consider integer solutions of the system of inequalities (**??**). Let us prove by contradiction that there are no more than 20 integer solutions to (**??**). Assume that $x_1 < x_2 < \ldots < x_{21}$ satisfy (**??**). Since the coefficients of $P(x)$ are integer, $P(x_{21}) - P(x_1)$ is divisible by $x_{21} - x_1 \geq 20$ and therefore $P(x_{21}) = P(x_1) = r$. Similarly $P(x_{21}) = P(x_i) = r$ for $i = 2, \dots, 10$ since $x_{21} - x_i \geq 11$. Also $P(x_1) = P(x_k) = r$ for $k = 12, \dots, 21$ since $x_k - x_1 \geq 11$. Therefore all x_j except x_{11} are roots of $P(x) - r$ and thus $P(x) = a(x - x_1) \cdots (x - x_{10})(x - x_{12}) \cdots (x - x_{21}) + r$. But then $|P(x_{11}) - r| \geq (10!)^2$ which is a contradiction.

5. There is a non-isosceles triangle. Peter and Basil play the following game. On each his turn Peter chooses a point in the plane. Basil responds by painting it into red or blue. Peter wins if some triangle similar to the original one has all vertices of the same colour. Find the minimal number of moves Peter needs to win no matter how Basil would play (independently of the shape of the given triangle).

ANSWER. 5.

SOLUTION. Peter selects triangle ABC (an original one). Without loss of generality, Basil paints A and B blue and C red. Then Peter selects D and E on the same side of AB as C so that triangles ABC, BDA and EAB are similar (with vertices in the matching order). Basil is forced to paint them red.



Now prove that triangle EDC and EAB are similar. Observe that $\angle DAE = \angle CBE$. Indeed, $\angle DAE = \angle DAB - \angle EAB$ and $\angle CBE = \angle ABE - \angle ABC$ and those angles are equal due to similarity. Also $BD : AB = AB : CA = EA : BE$.

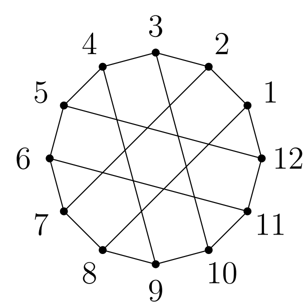
Thus triangles DAE and CBE are similar and in triangles EDC and EAB angles E are equal and $DE : CE = AE : BE$ and therefore they are similar.

Remark. This could be described using complex numbers terminology. Indeed, let us introduce coordinate system on the complex plane $\mathbb{C}$ such that points C, A and B correspond to complex numbers 0, z and z^2 respectively (it is always possible). Let us add points $w(E)$ and $wz(D)$ with $w = z^2 - z + 1$. Then triangles CAB, ABD, BEA CED are similar. Indeed triangles CAB and CED could be obtained from triangle Δ with vertices $(0, 1, z)$ by multiplication by z and w respectively; triangle BEA could be obtained from Δ by multiplication by $(1 - z)$ and shift by z^2, and triangle ABD could be obtained from Δ by multiplication by $z^2 - z$ and shift by z.

6. In some country every town has a unique number. In a flight directory for any two towns there is an indication whether or not they are connected by a direct non-stop flight. It is known that for any two assigned numbers M and N one can change the numeration of towns so that the town with number M gets the number N but the directory remains correct.

Is it always true that for any two assigned numbers M and N one can change the numeration of towns so that the towns with numbers M and N interchange their numbers but the directory is still correct?

Answer: No.
Solution. Observe that the figure drawn here is symmetric with respect to each diameter passing through the middle of the small chord. These symmetries allows us to interchange neighbouring towns and then several symmetries allows us to transfer any town into any other town.



Assume that we can exchange towns 1 and 3. Then their only common connected town 2 must remain on its place. Then its connected town 7 must also remain on its place. But 1 and 7 have two common connected towns (8 and 2) while 3 and 7 have only one common connected town (2).

Remark. Another example: tetrahedron with cut vertices. Then there is a graph with 12 vertices and 18 edges which has the same properties.

7. Consider a polynomial $P(x)$ such that

$P(0) = 1;\quad (P(x))^2 = 1 + x + x^{100}Q(x),$ where $Q(x)$ is also a polynomial.

Prove that in the polynomial $(P(x) + 1)^{100}$ the coefficient at x^{99} is zero.

Solution. Observe that $(P(x)+1)^{100}+(1-P(x))^{100}$ contains only even powers of $P(x)$ and therefore is a polynomial of degree 50 in $(P(x))^2$ i.e. of $(1+x)^{50}$ modulo polynomial divisible by x^{100}. However $1-P(x)$ is divisible by x and therefore $(1-P(x))^{100}$ is divisible by x^{100}.

Remark. More generally $(P(x)+1)^n$ is a polynomial in $(P(x))^2$ of degree $\lfloor n/2 \rfloor$ modulo polynomial divisible by x^n and therefore coefficients at x^m, $m = \lfloor n/2 \rfloor + 1, \ldots, n-1$ are zeros.